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FAST TRACK COMMUNICATION

An infinite family of solvable and integrable quantum systems on a plane

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Abstract
An infinite family of exactly solvable and integrable potentials on a plane is introduced. It is shown that all already known rational potentials with the above properties allowing separation of variables in polar coordinates are particular cases of this family. The underlying algebraic structure of the new potentials is revealed as well as its hidden algebra. We conjecture that all members of the family are also superintegrable and demonstrate this for the first few cases. A quasi-exactly-solvable and integrable generalization of the family is found.

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1. Introduction

Some quantum mechanical systems can be characterized by two differently defined global properties. The first has been called exact solvability and it means that all energy levels can be calculated algebraically and the corresponding wavefunctions can be obtained as polynomials in the appropriate variables, multiplied by some overall gauge factor. The other property is that of integrability, namely the existence of n integrals of motion that are well-defined quantum mechanical operators, commuting with the Hamiltonian and amongst each other.

A more restrictive property than integrability is superintegrability: the existence of more integrals of motion than degrees of freedom. A maximally superintegrable system has $2n - 1$ integrals of motion, including the Hamiltonian. Only subsets of n of them can commute amongst each other.

Any one-dimensional system is integrable and also maximally superintegrable by definition. In this communication, we concentrate on the two-dimensional case where the situation is quite different. A two-dimensional system is integrable if it allows two integrals of motion and maximally superintegrable if it allows three. Some time ago, it was conjectured that all maximally superintegrable systems for $n = 2$ are exactly solvable [1]. Here we will show that several exactly solvable systems are in fact maximally superintegrable. Though they seem very different, they are particular cases of a parametric family of Hamiltonians.

Let us consider the following Hamiltonian in $\mathbf{R}^2$ written in polar coordinates:

$$H_k(r,\varphi;\omega,\alpha,\beta) = -\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\varphi^2 + \omega^2 r^2 + \frac{\alpha k^2}{r^2\cos^2 k\varphi} + \frac{\beta k^2}{r^2 \sin^2 k\varphi}, \tag{1}$$

where $\alpha,\beta > -\frac{1}{4k^2}$, ω and $k \neq 0$ are parameters. For $k = 1$, this system was introduced in [2, 3] and has been called the Smorodinsky–Winternitz system [4]. For $k = 2$, the Hamiltonian (1) corresponds to the so-called rational BC_2 model [5, 6]. For $k = 3$, it describes the Wolfes model [7] (it is the rational G_2 model in the Hamiltonian reduction method nomenclature [5, 6]); if $\alpha = 0$, it reduces to the Calogero model [8]. The configuration space of (1) is given by the sector $\frac{\pi}{2k} \geqslant \varphi \geqslant 0$, $r \in [0,\infty)$ which is the Weyl chamber for BC_2 if $k = 2$ and G_2 if $k = 3$, respectively.

There is an interesting feature of the Hamiltonian (1) connecting different values of k, namely

$$H_{2\ell}(r,\varphi;\omega,0,\beta) = H_\ell(r,\varphi;\omega,\beta,\beta), \tag{2}$$

$$H_{2\ell}(r,\varphi;\omega,\alpha,0) = H_\ell\left(r,\varphi - \frac{\pi}{4\ell};\omega,\alpha,\alpha\right). \tag{3}$$

The Hamiltonian (1) to our knowledge includes *all* published superintegrable systems in a Euclidean plane E_2 that allow the separation of variables in polar coordinates.

2. Exact solvability

It is well known that the model (1) for $k = 1, 2, 3$ is exactly solvable (the energies and eigenfunctions can be found explicitly). For $\alpha = 0$ and k integer, the exact solvability of the Hamiltonian (1) was mentioned in [9]. It can be immediately checked by a direct calculation that the ground state of (1) is given by

$$\Psi_0 = r^{(a+b)k}\cos^a k\varphi \sin^b k\varphi\, \mathrm{e}^{-\frac{\omega r^2}{2}}, \qquad E_0 = 2\omega[(a+b)k+1], \tag{4}$$

where $\alpha = a(a-1)$ and $\beta = b(b-1)$. If we make a gauge rotation of the Hamiltonian (1),

$$h_k = \Psi_0^{-1}(H_k - E_0)\Psi_0, \tag{5}$$

we obtain the operator

$$h_k = -\partial_r^2 + \left(2\omega r - \frac{2k(a+b)+1}{r}\right)\partial_r - \frac{1}{r^2}\partial_\varphi^2 - \frac{2k}{r^2}(-a\tan k\varphi + b\cot k\varphi)\partial_\varphi, \tag{6}$$

for which the lowest eigenfunction is a constant with zero eigenvalue.

The original eigenfunctions $\Psi(r,\varphi)$ of the Hamiltonian (1) are related to those of the transformed Hamiltonian h_k as follows: $\Psi(r,\varphi) = \Psi_0(r,\varphi)\Xi(r,\varphi)$. Let us solve the original problem (1) in a traditional way by a separation of variables in h_k. Thus, we assume $\Xi(r,\varphi) = R(r)\Phi(\varphi)$ and write

$$h_k = h_r + \frac{1}{r^2}h_\varphi. \tag{7}$$

The operator h_φ written in the new coordinate $z = \sin^2 k\varphi$ reads as

$$h_\varphi = 4k^2 z(z-1)\partial_z^2 + 4k^2\left[(a+b+1)z - b - \tfrac{1}{2}\right]\partial_z. \tag{8}$$

The eigenvalue problem $h_\varphi \Phi = \Lambda_n \Phi$, where Λ_n is the separation constant, has polynomial eigenfunctions

$$\Phi_n(z) = P_n^{(a-1/2,b-1/2)}(2z-1), \qquad \Lambda_n = 4k^2 n(n+a+b), \qquad n = 0, 1, 2, \ldots, \tag{9}$$

where $P_n^{(a-1/2,b-1/2)}(2z-1)$ is a Jacobi polynomial. Now the eigenvalue problem for the operator

$$h_k = -\partial_r^2 + \left(2\omega r - \frac{2ak+2bk+1}{r}\right)\partial_r + \frac{\Lambda_n}{r^2} \tag{10}$$

appears. Let us perform a further gauge rotation of h_r:

$$\tilde{h}_k = r^{-\gamma} h_k r^{\gamma} = -\partial_r^2 + \left(2\omega r - \frac{2ak+2bk+2\gamma+1}{r}\right)\partial_r + \frac{\Lambda_n - 2(a+b)k\gamma - \gamma^2}{r^2} + 2\omega\gamma. \tag{11}$$

We absorb the term $2\omega\gamma$ in the energy and choose $\gamma = 2kn$ so as to remove the $1/r^2$ term:

$$\gamma^2 + 2(a+b)k\gamma - \Lambda_n = 0.$$

The resulting radial operator in the $t = r^2$ variable

$$\tilde{h}_r = -4t\partial_t^2 + 4[\omega t - k(2n+a+b) - 1]\partial_t \tag{12}$$

has the eigenstates

$$R_N(t) = L_N^{(k(2n+a+b))}(\omega t), \qquad E_N = 4\omega N, \tag{13}$$

where $L_N^{(k(2n+a+b))}(\omega t)$ is a Laguerre polynomial. Finally, the eigenstates of (1) are

$$\begin{aligned} \Psi_{N,n} &= r^{2nk} R_N(r^2) P_n^{(a-1/2,b-1/2)}(2\sin^2 k\varphi - 1)\Psi_0, \\ E_{N,n} &= 2\omega[2N + (2n+a+b)k + 1]. \end{aligned} \tag{14}$$

All formulae remain valid for any real $k \neq 0$. In particular, both $R_N(r^2)$ and $P_n^{(\alpha,\beta)}(z)$ remain polynomials. The eigenvalues are linear in the quantum numbers N, n. For integer (and rational) k there is a degeneracy of states, which is determined by the number of solutions of the equation

$$N + kn = \text{integer}.$$

Varying k, we can change degeneracy leaving the spectra linear in the quantum numbers N, n.

If k in (1) takes integer values, we have a Lie-algebraic interpretation of the problem (1). In order to reveal it, let us make the following change of variables:

$$t = r^2, \qquad u = r^{2k}\sin^2 k\varphi. \tag{15}$$

The resulting gauge-transformed Hamiltonian (1) in these coordinates takes an algebraic form:

$$\begin{aligned} h_k = &-4t\partial_t^2 - 8ku\partial_{tu}^2 - 4k^2 t^{k-1} u\partial_u^2 \\ &+ 4[\omega t - (a+b)k - 1]\partial_t + [4\omega k u - 2k^2(2b+1)t^{k-1}]\partial_u. \end{aligned} \tag{16}$$

It coincides with already known expressions for the Hamiltonian for $k = 1, 2, 3$ in appropriate variables (see [1, 6]). What is its underlying hidden algebra if any?

The Hamiltonian h_k preserves the space of polynomials

$$\mathcal{P}_{\mathcal{N}}^{(s)} = (t^p u^q | 0 \leqslant (p+sq) \leqslant \mathcal{N}), \qquad \mathcal{N} = 0, 1, 2, \ldots, \tag{17}$$

for $s \geqslant k-1$ and any integer $\mathcal{N}$. Hence, it has infinitely many finite-dimensional invariant subspaces $\mathcal{P}_{\mathcal{N}}^{(s)}$. These spaces can be ordered forming an *infinite flag*,

$$\mathcal{P}_0^{(s)} \subset \mathcal{P}_1^{(s)} \subset \mathcal{P}_2^{(s)} \cdots \mathcal{P}_{\mathcal{N}}^{(s)} \cdots \tag{18}$$

for fixed s. We call this flag $\mathcal{P}^{(s)}$. The space $\mathcal{P}_{\mathcal{N}}^{(s)}$ is a finite-dimensional irreducible representation space of the infinite-dimensional finitely generated Lie algebra $g^{(s)} \supset gl(2,\mathbf{R}) \ltimes \mathbf{R}^{s+1} \oplus T_s$ of monomials in $(s+6)$-generating operators. These generating operators are [10] (see also [11, 12])

$$\begin{aligned}
&J^1 = \partial_t,\\
&J^2_{\mathcal{N}} = t\partial_t - \frac{\mathcal{N}}{3}, \qquad J^3_{\mathcal{N}} = su\partial_u - \frac{\mathcal{N}}{3},\\
&J^4_{\mathcal{N}} = t^2\partial_t + stu\partial_u - \mathcal{N}t,\\
&R_i = t^i\partial_u, \qquad i = 0, 1, \ldots, s,
\end{aligned} \tag{19}$$

and

$$T_s = u\partial_t^s \tag{20}$$

(see [6]). The generator $J^3_{\mathcal{N}}$ is the central generator of the $gl(2,\mathbf{R})$-algebra. The generators (19) of the non-semisimple Lie algebra $gl(2) \ltimes R^{s+1}$ are vector fields on line bundles over an s-Hirzebruch surface [12]. The meaning of the generator (20) for $s > 1$ is unclear.

For $s = 1$, the algebra $g^{(1)}$ coincides with the algebra $sl(3)$. It has the space (17) for $s = 1$ as an invariant subspace and acts irreducibly there. It is important to note that the space $\mathcal{P}_{\mathcal{N}}^{(s)}$ is a finite-dimensional (reducible) representation space of the finite-dimensional non-semisimple Lie algebra $gl(2,\mathbf{R}) \ltimes \mathbf{R}^{s+1}$ (see [6]):

$$\tilde{\mathcal{P}}^{(s)}_{\mathcal{N},p} = \langle t^{n_1}u^{n_2} | 0 \leqslant (n_1 + sn_2) \leqslant \mathcal{N} \text{ and } 0 \leqslant n_2 \leqslant p\rangle. \tag{21}$$

For fixed s and p, these spaces form the flag $\tilde{\mathcal{P}}^{(s)}_p$. Each such flag for $s \geqslant k-1$ is preserved by the Hamiltonian h_k. This gives information about the structure of the eigenfunctions. In particular, it implies the existence of a family of eigenfunctions which depend on the variable t only.

It can be immediately checked that h_k for fixed integer k preserves the flag $\mathcal{P}^{(s)}$ for $s = k-1$, $s = k$ or $s > k$ assuming the hidden algebras $g^{(k-1)}$, $g^{(k)}$, $g^{(s)}$, respectively. It is worth mentioning that the first case $s = k-1$ supports the already known hidden algebras of the trigonometric BC_2 for $k = 2$ and G_2 for $k = 3$ models, respectively, in contrast to the second or third case. However, later on we will see that the case $s = k-1$ is excluded (see section 3).

The fact that s can take any values $s \geqslant k-1$ reflects a degeneracy of eigenstates of the original problem (1). For particular cases $k = 2, 3$, it was already mentioned in the paper [6]. This degeneracy is removed by the algebraic form of the integrals of motion (see below). Hence, for any integer k the algebraic Hamiltonian (16) can be rewritten in terms of the generators (19) (without the operator $J^4_{\mathcal{N}}$) (see theorem 4.3 from [11]):

$$\begin{aligned}
h_k = {} & -4J^2J^1 - 8J^3J^1 - 4kR_{k-1}J^3\\
& + 4\omega J^2 - 4[(a+b)k-1]J^1 + 4\omega J^3 - 2k^2(2b+1)R_{k-1},
\end{aligned} \tag{22}$$

where $J^i \equiv J^i_0$.

3. Complete Integrability

It is obvious that

$$\mathcal{X}_k(\alpha,\beta) = -L_3^2 + \frac{\alpha k^2}{\cos^2 k\varphi} + \frac{\beta k^2}{\sin^2 k\varphi}, \tag{23}$$

where $L_3 = \partial_\varphi$ is the 2D angular momentum, is an integral of motion [2, 3]. Its existence is directly related to the separation of variables in polar coordinates in the Schroedinger equation for (1). Therefore, the Hamiltonian (1) defines a completely integrable system for any real $k \neq 0$ which is also exactly solvable.

After a gauge rotation $x_k = \Psi_0^{-1}(\mathcal{X}_k - c_k)\Psi_0$, this integral takes the algebraic form

$$x_k = -4k^2u(t^k - u)\partial_u^2 - 4k^2\left[\left(b + \tfrac{1}{2}\right)t^k - (a+b+1)u\right]\partial_u, \tag{24}$$

where $c_k = k^2(a+b)^2$ is the lowest eigenvalue of the integral $\mathcal{X}_k$. It can be easily checked that x_k has infinitely many finite-dimensional invariant subspaces: it preserves the flag $\mathcal{P}^{(s)}$ for any $s \geqslant k$. The integral x_k can be rewritten in the generators (19) as

$$x_k = -4kJ^3R_k + 4J^3J^3 - 4k^2\left(b + \tfrac{1}{2}\right)R_k + 4k(a+b)J^3. \tag{25}$$

The presence of the generator R_k excludes the algebra (19) and (20) for $s = k - 1$ as hidden algebra (see the above discussion). It indicates that the hidden algebra of (23) is $g^{(s)}$ for $s \geqslant k$. Hence, the hidden algebra of the quantum system (1) with the integral (23) is $g^{(s)}$ with $s \geqslant k$.

4. Superintegrability

The next question is the existence of an additional integral of motion $\mathcal{Y}_{2k}$ (presumably of order $2k$) for all integer values of k. If such an integral exists, then the system (1) is (maximally) superintegrable. For $k = 1$, the Smorodinsky–Winternitz system, this integral $\mathcal{Y}_2$ was found long ago (see [1, 2, 3]). It turned out to be a second-order differential operator. For $k = 2$ and $\omega = 0$, which is the case of the so-called singular rational BC_2 model, the integral $\mathcal{Y}_4$ was found by Olshanetsky–Perelomov in the representation theory approach [5]. This integral is a fourth-order differential operator. For $k = 3$ and $\omega = \alpha = 0$ (the so-called singular Calogero model), the corresponding integral is a third-order differential operator [5]. For $\alpha \neq 0$ (the so-called singular Wolfes model), it was mentioned in [5] that it has to be of the sixth order. For the general Wolfes model $\omega \neq 0$ (the rational G_2 model in the Hamiltonian reduction method nomenclature), Quesne [13] found this integral explicitly in the Dunkl operator formalism. The integral $\mathcal{Y}_6$ is a sixth-order differential operator.

If the integral $\mathcal{Y}_{2k}$ exists, it should have the same eigenfunctions as H_k. Hence by a gauge rotation and change of variables (15), we can obtain an operator y_{2k}:

$$y_{2k} = \Psi_0^{-1}(\mathcal{Y}_{2k} - C_{2k})\Psi_0|_{t,u}, \tag{26}$$

such that y_{2k} is a differential operator of some order in t and u with polynomial coefficients and C_{2k} is the lowest eigenvalue of the integral $\mathcal{Y}_{2k}$. The described algebraic form y_{2k} would be a consequence of the fact that both h_k (16) and y_{2k} should preserve the same flag of polynomials.

For the case $k = 1$, the integral $\mathcal{Y}_2$ was found in [3]. In Cartesian coordinates, $\mathcal{Y}_2$ is of second order and it can be written as

$$\mathcal{Y}_2 = \partial_x^2 - \omega^2x^2 - \frac{\alpha}{x^2}. \tag{27}$$

The algebraic form of the integral was calculated in [1]. In the coordinates (15), the integral (27) is

$$\frac{y_2}{4} = (t-u)\partial_t^2 + \left[\omega(u-t) + a + \frac{1}{2}\right]\partial_t, \tag{28}$$

where the constant $C_2 = -\omega(2a+1)$. The integral y_2 (28) contains the term $u\partial_t$ which is present in the algebra $g^{(1)}$ (see (20)). It indicates unambiguously that for the case $k = 1$, the hidden algebra should correspond to $s = 1$. Hence, there is no ambiguity for the $k = 1$ case. The hidden algebra is fixed and it is $g^{(1)} \equiv sl(3)$ which is generated by $gl(2) \ltimes R^2 \oplus T_1 \subset gl(3)$. The Lie algebraic form of y_2 is as follows:

$$\frac{y_2}{4} = J^2J^1 - T_1J^1 + \omega T_1 - \omega J^2 + \left(a + \frac{1}{2}\right)J^1. \tag{29}$$

We stress that the generator T_1 (see (20)) appears explicitly in (29).

For the case $k = 2$ (the rational BC_2 model), we find the higher integral $\mathcal{Y}_4$ explicitly:

$$\mathcal{Y}_4 = \left(\partial_x^2 - \omega^2 x^2 - \partial_y^2 + \omega^2 y^2\right)^2 + \left\{\partial_x^2, \frac{(x^2-y^2)\beta}{x^2y^2} - \frac{4(x^2+y^2)\alpha}{(x^2-y^2)^2}\right\}$$
$$+ \left\{\partial_x\partial_y, -\frac{16xy\alpha}{(x^2-y^2)^2}\right\} + \left\{\partial_y^2, -\frac{(x^2-y^2)\beta}{x^2y^2} - \frac{4(x^2+y^2)\alpha}{(x^2-y^2)^2}\right\}$$
$$+ \frac{16\alpha^2}{(x^2-y^2)^2} + \frac{(x^2-y^2)^2\beta^2}{x^4y^4} + \frac{8\alpha\beta}{x^2y^2} - \frac{2(x^4+y^4)\beta\omega^2}{x^2y^2}, \tag{30}$$

where $\{,\}$ denotes an anticommutator. Making the gauge rotation $\Psi_0^{-1}(\mathcal{Y}_4 - C_4)\Psi_0$ and the change of variables (15), we arrive at the algebraic form of the integral

$$\begin{aligned}\frac{y_4}{16} = {}& (t^2-u)\partial_t^4 - 8(t^2-u)u\partial_t^2\partial_u^2 + 16(t^2-u)u^2\partial_u^4 - 2[\omega t^2 - (2a+1)t - \omega u]\partial_t^3\\
& - 4[(2b+1)t^2 - 2(a+b+1)u]\partial_t^2\partial_u + 8u[\omega t^2 - (2a+1)t - \omega u]\partial_t\partial_u^2\\
& + 16u[(2b+3)t^2 - 2(a+b+2)u]\partial_u^3\\
& + 16[\omega^2t^2 - 3(2a+1)\omega t - \omega^2 u + (2a+1)(2a+2b+1)]\partial_t^2\\
& - 4[(2b+1)\omega t^2 - (2a+1)(2b+1)t - 2(a+b+1)\omega u]\partial_t\partial_u\\
& + 4[(2b+1)(2b+3)t^2 + (2a+1)\omega tu - 2(2a^2+6ab+2b^2+8a+7b+5)u]\partial_u^2\\
& + \omega(2a+1)(\omega t - 2a - 2b - 1)\partial_t + 2(2a+1)(2b+1)(\omega t - 2a - 2b - 1)\partial_u,\end{aligned} \tag{31}$$

where $C_4 = 4\omega^2[2a(a+1) - b(b-1)]$. The two terms $t^2\partial_u$ and $u\partial_t^2$ in y_4 imply $s = 2$. Hence, the hidden algebra for $k = 2$ is $g^{(2)}$. The Lie-algebraic form of y_4 is as follows:

$$\begin{aligned}\frac{y_4}{16} = {}& J^2J^2J^1J^1 - J^1J^1T_2 + 2J^1J^1J^3J^3 + 4J^3J^3R_2R_0 - 4J^2J^2J^3R_0 - 2J^3J^3J^3R_0\\
& - 2\omega J^2J^2J^1 - 2\omega J^3J^3J^1 + 2(2a+1)J^2J^1J^1 - 4(2b+1)J^2J^2R_0\\
& + 4\omega J^3R_2J^1 - 4(2a+1)J^3R_1J^1 + 8(2b+3)J^3R_1R_1 - 8(a+b+2)J^3J^3R_0\\
& + (2a+1)(2a+2b+1)J^1J^1 - 3\omega(2a+1)J^2J^1 + \omega^2J^2J^2 + 4\omega(a+b+1)J^3J^1\\
& - 4\omega(2b+1)R_2J^1 + 64(2a+1)(2b+1)J^2R_0 + 2\omega(2a+1)J^3R_1\\
& - 4(2a^2+6ab+2b^2+8a+7b+5)J^3R_0\\
& + 4(2b+1)(2b+3)R_2R_0 + 2\omega T_2J^1 + 8(a+b+1)T_2R_0\\
& - (2a+1)(2a+2b+1)J^1 + \omega^2(2a+1)J^2\\
& + 2\omega(2a+1)(2b+1)R_1 - 2(2a+1)(2b+1)(2a+2b+1)R_0 - \omega^2T_2.\end{aligned} \tag{32}$$

The generator T_2 (see (20)) again appears explicitly in (32).

For the case $k = 3$ (the rational G_2 model), we find the higher integral $\mathcal{Y}_6$ explicitly by a straightforward (brute force) calculation. It is of sixth order (see [14, appendix A]). Its lowest eigenvalue is

$$C_6 = 4\omega^3(3a+3b+1)(5a^2+36ab-27b^2+a+45b+4). \tag{33}$$

Making the gauge rotation $\Psi_0^{-1}(\mathcal{Y}_6 - C_6)\Psi_0$ and the change of variables (15), we arrive at the algebraic form of the integral y_6 (see [14, appendix A]). The two elements $R_3 = t^3\partial_u$ and $T_3 = u\partial_t^3$ are present in y_6 and unambiguously point to $s = 3$. Hence, the hidden algebra of the model at $k = 3$ is $g^{(3)}$.

For the case $k = 4$, we again find the higher integral $\mathcal{Y}_8$ explicitly by a brute-force calculation as an eight-order differential operator (see [14, appendix B]). Making the gauge rotation $\Psi_0^{-1}(\mathcal{Y}_8 - C_8)\Psi_0$ and the change of variables with

$$C_8 = 4\omega^4[3200a^4 + 512a^3(31b + 10) + 16a^2(206b + 159)(2b - 3) + 16a(310b^3 - 187b^2 - 443b + 105) + 1133b^4 + 150b^3 - 176b^2 + 493b + 4], \tag{34}$$

we arrive at the algebraic form of the integral, y_8 (see [14, appendix B]). The elements $R_4 = t^4\partial_u$ and $T_4 = u\partial_t^4$ in y_8 imply $s = 4$. The hidden algebra of the model for $k = 4$ is $g^{(4)}$ which contains the generator T_4.

We were unable to prove the existence of the higher order integrals $\mathcal{Y}_{2k}$ for integer k with $k > 4$ due to the fast growing complexity of the brute-force calculations. However, we feel justified in formulating the following conjecture.

Conjecture. *An integral of motion $\mathcal{Y}_{2k}$ of order $2k$ exists for the Hamiltonian (1) for all positive integer values of k. In Cartesian coordinates, $\mathcal{Y}_{2k}$ is a differential operator of order $2k$ with rational coefficients. The gauge transformation (26) together with the change of variables (15) transforms $\mathcal{Y}_{2k}$ into the algebraic operator y_{2k} that has polynomial coefficients. The integral y_{2k} is an element of order $2k$ in the enveloping algebra of the hidden algebra $g^{(k)}$. In particular, y_{2k} contains the terms $4^k[(J^1)^k - T_k](J^1)^k$ which fix $k = s$ in the hidden algebra (19) and (20). In the limit $\omega = \alpha = 0$, the operator $\mathcal{Y}_{2k}(0, 0, \beta)$ is reduced to the square of an operator of order k.*

Our conjecture is based on the fact that the gauge-rotated Hamiltonian h_k (5) preserves the flag of polynomials (17) as do all the elements of the underlying hidden algebra $g^{(k)}$ (19) and (20). All aspects of this conjecture have been confirmed for $k = 1, 2, 3$ and 4 for general ω, α, β as well as for $k = 1, \ldots, 6, 8$ for $\omega = \alpha = 0, \beta \neq 0$. The consideration of $k > 4$ for general ω, α, β requires a different approach other than the brute-force one. A proof of the conjecture could be based on a direct analysis of the commutation relations of the hidden algebra (19) and (20).

Any operator preserving this flag must lie in the enveloping algebra of $g^{(k)}$ for given k. The gauge-rotated integrals y_{2k} must hence have an algebraic form for all k, as exemplified by $k = 1, 2, 3$ and 4.

The form of the integrals $\mathcal{Y}_{2k}$ is not unique since we can modify it by adding polynomials in the Hamiltonian (1) and integral $\mathcal{X}_k$ (23). Our convention is to require that the highest order terms in $\mathcal{Y}_{2k}$ should have the form

$$[\mathrm{Re}(\partial_1 + \mathrm{i}\partial_2)^k]^2. \tag{35}$$

The lower order terms in $\mathcal{Y}_{2k}$ could be further simplified by linear combinations with lower order polynomials in (1) and (23). We also require that $\mathcal{Y}_{2k}$ be a Hermitian operator, and this implies that it will contain only even powers of the derivatives $\left(\partial_1^m\partial_2^n, m+n = 0, 2, 4, \ldots, 2k\right)$.

5. A quasi-exactly-solvable extension

Some years ago, a new class of the Schroedinger equations was discovered for which a finite number of eigenstates can be calculated by purely algebraic means. They were called *quasi-exactly-solvable* [15, 16]. These problems occupy an intermediate place between exactly solvable problems and non-solvable ones. A large body of articles dedicated to these problems was published during the last 20 years. The articles have ranged from various branches of physics to pure mathematics.

Surprisingly, there exists a quasi-exactly-solvable generalization of the Hamiltonian (1):

$$H^{(\mathrm{qes})}_{k,\mathcal{N}}(r,\varphi;\omega,\alpha,\beta) = -\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\varphi^2 + \lambda^2 r^6 + 2\lambda\omega r^4 + [\omega^2 - 2\lambda(2\mathcal{N}+2+k(a+b))]r^2 + \frac{\alpha k^2}{r^2\cos^2 k\varphi} + \frac{\beta k^2}{r^2\sin^2 k\varphi}, \tag{36}$$

(cf [15–17]), where $\dim\mathcal{P}^{(k)}_{\mathcal{N}} \approx \frac{\mathcal{N}^2}{2k} + 1$ eigenstates can be found explicitly (algebraically). These algebraic eigenfunctions have the form of a polynomials $p(t,u)$ from the space $\mathcal{P}^{(k)}_{\mathcal{N}}$ (17) multiplied by a factor $\Psi_0^{(\mathrm{qes})}$:

$$\Psi_0^{(\mathrm{qes})} = r^{(a+b)k}\cos^a k\varphi \sin^b k\varphi\, \mathrm{e}^{-\frac{\omega r^2}{2} - \frac{\lambda r^4}{4}}, \tag{37}$$

namely

$$\Psi_{alg}^{(\mathrm{qes})} = p(t,u)\Psi_0^{(\mathrm{qes})}. \tag{38}$$

Hence, the number of algebraic states is equal to the dimension of the space $\mathcal{P}^{(k)}_{\mathcal{N}}$.

The gauge-rotated Hamiltonian (36),

$$h^{(\mathrm{qes})}_{k,\mathcal{N}} = -\left(\Psi_0^{(\mathrm{qes})}\right)^{-1}\left(H^{(\mathrm{qes})}_{k,\mathcal{N}} - E_0\right)\Psi_0^{(\mathrm{qes})},$$

where E_0 is some parameter, in the variables (15) has the algebraic form

$$h^{(\mathrm{qes})}_{k,\mathcal{N}} = 4t\partial_t^2 + 8ku\partial_{tu}^2 + 4k^2t^{k-1}u\partial_u^2 + 4[\lambda t^2 - \omega t + (a+b)k + 1]\partial_t + [4\lambda ktu - 4\omega ku + 2k^2(2b+1)t^{k-1}]\partial_u - 4\lambda\mathcal{N}t. \tag{39}$$

It is easy to check that (39) preserves the space $\mathcal{P}^{(k)}_{\mathcal{N}}$ (17). Hence, it can be rewritten in generators of the algebra (19), $gl(2) \ltimes R^{k+1}$ [11] and indeed we have

$$h^{(\mathrm{qes})}_{k,\mathcal{N}}/4 = \left(J^2_{\mathcal{N}} + 2J^3_{\mathcal{N}}\right)J^1 + kJ^3_{\mathcal{N}}R_{k-1} + [(a+b)k + 1 + \mathcal{N}]J^1 - \omega\left(J^2_{\mathcal{N}} + J^3_{\mathcal{N}}\right) - \lambda J^4_{\mathcal{N}} + \frac{k}{6}[2\mathcal{N} + 3k(2b+1)]R_{k-1}. \tag{40}$$

Evidently, the QES problem is completely integrable: $\mathcal{X}_k$ (see (23)) commutes with (36). The algebraic form of $\mathcal{X}_k$ after a gauge rotation with (37) in variables (t,u) remains the same (24). The Lie-algebraic form (25) is slightly modified:

$$x_k = -4kJ^3_{\mathcal{N}}R_k + 4J^3_{\mathcal{N}}J^3_{\mathcal{N}} - 4k\left[k\left(b+\frac{1}{2}\right) + \frac{\mathcal{N}}{3}\right]R_k + 4\left[k(a+b) + \frac{2\mathcal{N}}{3}\right]J^3_{\mathcal{N}} + \frac{4\mathcal{N}^2}{9}. \tag{41}$$

The question of the existence of a second integral and thus of the superintegrability of the Hamiltonian (36) remains open.

6. Conclusions

We have restricted this communication to the case of a Schroedinger equation in a two-dimensional Euclidean space E_2 and to the Hamiltonians allowing separation of variables in polar coordinates. The feature underlying the exact solvability, the complete integrability and the conjecture of maximal superintegrability is the existence of a hidden Lie algebra of differential operators. All elements of the hidden algebra and hence also of its enveloping algebra preserve an infinite flag of finite-dimensional subspaces of the space of wavefunctions.

The Hamiltonians and the integrals of motion of the entire family (1) considered in this paper are also elements of the enveloping algebra of $g^{(k)}$. The family contains *all* currently known superintegrable systems in E_2 that are separable in polar coordinates. It would be important to clarify whether the Hamiltonian (1) can be obtained by a Hamiltonian reduction procedure. This is the case for $k = 1, 2, 3$.

The first problem that remains open is to prove our conjecture, namely that the Hamiltonian (1) is superintegrable for all integer values of k. Another important question is that of the classical limit of the system with Hamiltonian (1). For $k = 1, 2$ and 3, these systems are all superintegrable. Chanu *et al* [18] have considered the classical case for $\omega = \alpha = 0$ and $k = 2n + 1$ and have conjectured that it is superintegrable for all integer n. We think that the classical limit of (1) is actually superintegrable for all values of ω, α and k. We plan to verify this conjecture directly by calculating the trajectories for the classical systems. If the systems are (maximally) superintegrable, then all bounded trajectories must be closed and the motion must be periodic [19].

The direct construction of the higher order integrals $\mathcal{Y}_{2k}$ for $k \geqslant 5$ seems intractable. More promising approaches would either involve an efficient use of the hidden algebra $g^{(k)}$ or possibly the use of Dunkl operators [20] as suggested for the Calogero model in [21] and for the Wolfes model in [13].

The close relation between exact solvability and maximal superintegrability has also been exemplified in n dimensions [22–24]. A very complete review of quantum completely integrable systems in n dimensions was recently given by Oshima [25]. For some cases, these systems are known to be exactly solvable. It would be of great interest to investigate their possible (maximal) superintegrability.

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